

# NON FORKING GOOD FRAMES WITHOUT LOCAL CHARACTER

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**ABSTRACT.** We continue [Sh:h].II, studying stability theory for abstract elementary classes. In [Sh E46], Shelah obtained a non-forking relation for an AEC,  $(K, \preceq)$ , with  $LST$ -number at most  $\lambda$ , which is categorical in  $\lambda$  and  $\lambda^+$  and has less than  $2^{\lambda^+}$  models of cardinality  $\lambda^{++}$ , but at least one. This non-forking relation satisfies the main properties of the non-forking relation on stable first order theories, but only a weak version of the local character.

Here, we improve this non-forking relation such that it satisfies the local character, too. Therefore it satisfies the main properties of the non-forking relation on superstable first order theories.

Using results of [Sh:h].II, we conclude that the function  $\lambda \rightarrow I(\lambda, K)$ , which assigns to each cardinal  $\lambda$ , the number of models in  $K$  of cardinality  $\lambda$ , is not arbitrary.

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## 1. PRELIMINARIES

Familiarity with AEC's is assumed.

*Hypothesis 1.1.*

- (1)  $(K, \preceq)$  is an AEC.
- (2)  $\lambda$  is a cardinal.
- (3) The Lowenheim Skolem Tarski number of  $(K, \preceq)$ ,  $LST(K, \preceq)$ , is at most  $\lambda$ .

**Definition 1.2.** Suppose  $M_0 \prec N$  in  $K_\lambda$ . We say that  $N$  is *universal* over  $M_0$  if for every  $M_1 \succ M_0$ , there is an embedding of  $M_1$  into  $N$  over  $M_0$ , namely, that fixes  $M_0$ .

The following proposition is a version of Fodor's Lemma (there is no mathematical reason to choose this version, but we think that it is convenient).

**Proposition 1.3.** *There exist no  $\langle M_\alpha : \alpha \in \lambda^+ \rangle$ ,  $\langle N_\alpha : \alpha \in \lambda^+ \rangle$ ,  $\langle f_\alpha : \alpha \in \lambda^+ \rangle$ ,  $S$  such that the following conditions are satisfied:*

- (1) *The sequences  $\langle M_\alpha : \alpha \in \lambda^+ \rangle$ ,  $\langle N_\alpha : \alpha \in \lambda^+ \rangle$  are  $\preceq$ -increasing continuous sequences of models in  $K_\lambda$ .*
- (2) *For every  $\alpha < \lambda^+$ ,  $f_\alpha : M_\alpha \rightarrow N_\alpha$  is a  $\preceq$ -embedding.*
- (3)  *$\langle f_\alpha : \alpha \in \lambda^+ \rangle$  is an increasing continuous sequence.*
- (4)  *$S$  is a stationary subset of  $\lambda^+$ .*
- (5) *For every  $\alpha \in S$ , there is  $a \in M_{\alpha+1} - M_\alpha$  such that  $f_{\alpha+1}(a) \in N_\alpha$ .*

*Proof.* Suppose there are such sequences. Denote  $M = \bigcup \{f_\alpha[M_\alpha] : \alpha \in \lambda^+\}$ . By clauses (4),(5),  $\|M\| = K_{\lambda^+}$ .  $\langle f_\alpha[M_\alpha] : \alpha \in \lambda^+ \rangle$ ,  $\langle N_\alpha \cap M : \alpha \in \lambda^+ \rangle$  are filtrations of  $M$ . So they are equal on a club of  $\lambda^+$ . Hence there is  $\alpha \in S$  such that  $f_\alpha[M_\alpha] = N_\alpha \cap M$ . Hence  $f_\alpha[M_\alpha] \subseteq N_\alpha \cap f_{\alpha+1}[M_{\alpha+1}] \subseteq N_\alpha \cap M = f_\alpha[M_\alpha]$  and so this is a chain of equivalences. Especially  $f_{\alpha+1}[M_{\alpha+1} \cap N_\alpha] = f_\alpha[M_\alpha]$ , in contradiction to condition (5).  $\dashv$

## 2. NON-FORKING FRAMES

The following definition, Definition 2.1 is an axiomatization of the non-forking relation in a superstable first order theory. If we subtract axiom 2.1(3)(c), we get the basic properties of the non-forking relation in  $(K_\lambda, \preceq \upharpoonright K_\lambda)$  where  $(K, \preceq)$  is stable in  $\lambda$ .

Sometimes we do not find a natural independence relation with respect to all the types. So first we extend the notion of an AEC in  $\lambda$  by adding a new function  $S^{bs}$  which assigns a collection of basic (because they are basic for our construction) types to each model in  $K_\lambda$ , and then add an independence relation  $\mathbb{U}$  on basic types.

We do not assume *the amalgamation property* in general, but we assume the amalgamation property in  $(K_\lambda, \preceq \upharpoonright K_\lambda)$ . This is a reasonable assumption, because it is proved in [Sh:h].I, that if an AEC is categorical in  $\lambda$  and the amalgamation property fails in  $\lambda$ , then under a plausible set theoretic assumption, there are  $2^{\lambda^+}$  models in  $K_{\lambda^+}$ .

**Definition 2.1.**  $\mathfrak{s} = (K, \preceq, S^{bs}, \mathbb{U})$  is a *good  $\lambda$ -frame* if:

- (1)  $(K, \preceq)$  is an AEC in  $\lambda$ .
- (2) (a)  $(K, \preceq)$  satisfies the joint embedding property.  
 (b)  $(K, \preceq)$  satisfies the amalgamation property.  
 (c) There is no  $\preceq$ -maximal model in  $K$ .
- (3)  $S^{bs}$  is a function with domain  $K$ , which satisfies the following axioms:
  - (a)  $S^{bs}(M) \subseteq S^{na}(M) = \{ga - tp(a, M, N) : M \prec N \in K, a \in N - M\}$ .
  - (b)  $S^{bs}$  respects isomorphisms: if  $ga - tp(a, M, N) \in S^{bs}(M)$  and  $f : N \rightarrow N'$  is an isomorphism, then  $ga - tp(f(a), f[M], N') \in S^{bs}(f[M])$ .

- (c) Density of the basic types: if  $M, N \in K_\lambda$  and  $M \prec N$ , then there is  $a \in N - M$  such that  $ga - tp(a, M, N) \in S^{bs}(M)$ .
- (d) Basic stability: for every  $M \in K$ , the cardinality of  $S^{bs}(M)$  is  $\leq \lambda$ .
- (4) the relation  $\mathbb{U}$  satisfies the following axioms:
  - (a)  $\mathbb{U}$  is set of quadruples  $(M_0, M_1, a, M_3)$  where  $M_0, M_1, M_3 \in K$ ,  $a \in M_3 - M_1$  and for  $n = 0, 1$   $ga - tp(a, M_n, M_3) \in S^{bs}(M_n)$  and it respects isomorphisms: if  $\mathbb{U}(M_0, M_1, a, M_3)$  and  $f : M_3 \rightarrow M'_3$  is an isomorphism, then  $\mathbb{U}(f[M_0], f[M_1], f(a), M'_3)$ .
  - (b) Monotonicity: if  $M_0 \preceq M_0^* \preceq M_1^* \preceq M_1 \preceq M_3 \preceq M_3^*$ ,  $M_1^* \cup \{a\} \subseteq M_3^{**} \preceq M_3^*$ , then  $\mathbb{U}(M_0, M_1, a, M_3) \Rightarrow \mathbb{U}(M_0^*, M_1^*, a, M_3^{**})$ . From now on, ' $p \in S^{bs}(N)$  does not fork over  $M$ ' will be interpreted as 'for some  $a, N^+$  we have  $p = ga - tp(a, N, N^+)$  and  $\mathbb{U}(M, N, a, N^+)$ '. See Proposition 2.2.
  - (c) Local character: for every limit ordinal  $\delta < \lambda^+$  if  $\langle M_\alpha : \alpha \leq \delta \rangle$  is an increasing continuous sequence of models in  $K_\lambda$ , and  $ga - tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$ , then there is  $\alpha < \delta$  such that  $ga - tp(a, M_\delta, M_{\delta+1})$  does not fork over  $M_\alpha$ .
  - (d) Uniqueness of the non-forking extension: if  $M, N \in K$ ,  $M \preceq N$ ,  $p, q \in S^{bs}(N)$  do not fork over  $M$ , and  $p \upharpoonright M = q \upharpoonright M$ , then  $p = q$ .
  - (e) Symmetry: if  $M_0, M_1, M_3 \in K_\lambda$ ,  $M_0 \preceq M_1 \preceq M_3$ ,  $a_1 \in M_1$ ,  $ga - tp(a_1, M_0, M_3) \in S^{bs}(M_0)$ , and  $ga - tp(a_2, M_1, M_3)$  does not fork over  $M_0$ , then there are  $M_2, M_3^* \in K_\lambda$  such that  $a_2 \in M_2$ ,  $M_0 \preceq M_2 \preceq M_3^*$ ,  $M_3 \preceq M_3^*$ , and  $ga - tp(a_1, M_2, M_3^*)$  does not fork over  $M_0$ .
  - (f) Existence of non-forking extension: if  $M, N \in K$ ,  $p \in S^{bs}(M)$  and  $M \prec N$ , then there is a type  $q \in S^{bs}(N)$  such that  $q$  does not fork over  $M$  and  $q \upharpoonright M = p$ .
  - (g) Continuity: let  $\delta < \lambda^+$  and  $\langle M_\alpha : \alpha \leq \delta \rangle$  be an increasing continuous sequence of models in  $K$  and let  $p \in S(M_\delta)$ . If for every  $\alpha \in \delta$ ,  $p \upharpoonright M_\alpha$  does not fork over  $M_0$ , then  $p \in S^{bs}(M_\delta)$  and does not fork over  $M_0$ .

**Proposition 2.2.** *If  $\mathbb{U}(M_0, M_1, a, M_3)$  and the types  $ga - tp(b, M_1, M_3^*)$ ,  $ga - tp(a, M_1, M_3)$  are equal, then we have  $\mathbb{U}(M_0, M_1, a, M_3)$ .*

*Proof.* Since  $ga - tp(b, M_1, M_3^*) = ga - tp(a, M_1, M_3)$ , there is an amalgamation  $(id_{M_3}, f, M_3^{**})$  of  $M_3$  and  $M_3^*$  over  $M_1$  with  $f(b) = a$ . By Definition 2.1(3)(b) (monotonicity)  $\mathbb{U}(M_0, M_1, a, M_3^{**})$ . Using again Definition 2.1(3)(b), we get  $\mathbb{U}(M_0, M_1, a, f[M_3^*])$ . Therefore by Definition 2.1(3)(a),  $\mathbb{U}(M_0, M_1, a, M_3^*)$ .  $\dashv$

**Definition 2.3.**

- (1)  $\mathfrak{s} = (K, \preceq, S^{bs}, nf)$  is an *almost* good  $\lambda$ -frame if  $\mathfrak{s}$  satisfies the axioms of a good  $\lambda$ -frame except maybe local character, but  $\mathfrak{s}$  satisfies weak local character.
- (2)  $\mathfrak{s}$  satisfies *weak local character* when there is a 2-ary relation,  $\prec^*$  on  $K_\lambda$  which is included in  $\prec \upharpoonright K_\lambda$  such that:
  - (a) for each  $M_0 \in K_\lambda$  there is  $M_1 \in K_\lambda$  with  $M_0 \prec^* M_1$ ,
  - (b) if  $M_0 \prec^* M_1 \preceq M_2 \in K_\lambda$  then  $M_0 \prec^* M_2$ ,
  - (c) if  $\langle N_\alpha : \alpha < \delta + 1 \rangle$  is a  $\prec^*$ -increasing continuous sequence of models in  $K_\lambda$ , then for some  $a \in N_{\delta+1}$  and some ordinal  $\alpha < \delta$ ,  $p =: ga - tp(a, N_\delta, N_{\delta+1})$  is a basic type, which does not fork over  $N_\alpha$ .

In the following definition ‘na’ means non-algebraic.

**Definition 2.4.** We define a function  $S^{na}$  with domain  $K_\lambda$  by  $S^{na}(M) := \{ga - tp(a, M, N) : M \preceq N, a \in N - M\}$ .

**Definition 2.5.** Let  $\mathfrak{s}$  be an almost good  $\lambda$ -frame.  $\mathfrak{s}$  is *full* if  $S^{bs} = S^{na}$ .

The following theorem says that the stability property in  $\lambda$  is satisfied and presents sufficient conditions for a universal model. The stability in  $\lambda$  can actually derived from [JrSh 875, Theorem 2.20].

**Theorem 2.6.**

- (1) *Suppose:*
  - (a)  $\mathfrak{s}$  is an almost good  $\lambda$ -frame (so indirectly, we assume basic stability).
  - (b)  $\langle M_\alpha : \alpha \leq \lambda \rangle$  is an increasing continuous sequence of models in  $K_\lambda$ .
  - (c)  $M_{\alpha+1}$  realizes  $S^{bs}(M_\alpha)$ .
  - (d)  $M_\alpha \prec^* M_{\alpha+1}$ .
 Then  $M_\lambda$  is universal over  $M_0$ .
- (2) *There is a model in  $K_\lambda$  which is universal over  $\lambda$ .*
- (3) *For every  $M \in K_\lambda$ ,  $|S(M)| \leq \lambda$ .*

*Proof.* Obviously (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3). Why does (1) hold? We have to prove that letting  $M_0 \prec N$ ,  $N$  can be embedded in  $M_\lambda$  over  $M_0$ . Toward a contradiction assume that:

(\*) There is no an embedding from  $N$  into  $M_\lambda$  over  $M_0$ .

Let  $cd$  be a bijection from  $\lambda \times \lambda$  onto  $\lambda$ . Now we choose  $N_\alpha, A_\alpha, \langle a_{\alpha,\beta} : \beta < \lambda \rangle, f_\alpha$  by induction on  $\alpha$  such that:

- (1)  $N_0 = N$ ,  $f_0 = id_{M_0}$
- (2)  $\langle N_\alpha : \alpha < \lambda \rangle$  is an increasing continuous sequence of models in  $K_\lambda$ .
- (3)  $\langle f_\alpha : \alpha < \lambda \rangle$  is an increasing continuous sequence of functions.
- (4)  $f_\alpha : M_\alpha \hookrightarrow N_\alpha$  is an embedding.
- (5)  $N_\alpha = \{a_{\alpha,\beta} : \beta < \lambda\}$ .
- (6)  $A_\alpha = \{cd(\gamma, \beta) : \gamma \preceq \alpha, ga - tp(a_{\gamma,\beta}, f_\alpha[M_\alpha], N_\alpha) \in S^{bs}(f_\alpha[M_\alpha])\}$ .

(7)  $a_{\gamma,\beta} \in f_{\alpha+1}[M_{\alpha+1}]$  where  $(\gamma, \beta) = cd^{-1}(\text{Min}(A_\alpha))$ .

*Why can we carry out the induction?* For  $\alpha = 0$  or limit, there is no problem. Suppose we have chosen  $N_\alpha, A_\alpha, \langle a_{\alpha,\beta} : \beta < \lambda \rangle, f_\alpha$ . If  $f_\alpha[M_\alpha] = N_\alpha$ , then  $f_\alpha^{-1} \upharpoonright N_0$  is an embedding over  $M_0$ , in contradiction to (\*). Thus  $f_\alpha[M_\alpha] \neq N_\alpha$ . Therefore there is a type in  $S^{bs}(f_\alpha[M_\alpha])$  which  $N_\alpha$  realizes. Hence  $A_\alpha \neq \emptyset$ . So by the definition of a type, there is no problem to find  $N_{\alpha+1}, A_{\alpha+1}, \langle a_{\alpha+1,\beta} : \beta < \lambda \rangle, f_{\alpha+1}$ .

*Why is this enough?* Define  $N_\lambda := \bigcup \{N_\alpha : \alpha < \lambda\}$ ,  $f_\lambda := \bigcup \{f_\alpha : \alpha < \lambda\}$ . By smoothness,  $f_\lambda[M_\lambda] \preceq N_\lambda$ . But  $f_\lambda[M_\lambda] \neq N_\lambda$  (otherwise  $f_\lambda^{-1} \upharpoonright N_0$  is an embedding over  $M_0$ , in contradiction to (\*)). So by *weak local character*, there is  $c \in N_\lambda - f_\lambda[M_\lambda]$  and there is a  $\gamma \in \lambda$  such that  $ga - tp(c, f_\lambda[M_\lambda], N_\lambda)$  does not fork over  $f_\gamma[M_\gamma]$ . Without loss of generality,  $c \in N_\gamma$ , because we can increase  $\gamma$ . Therefore there is  $\beta \in \lambda$  such that  $c = a_{\gamma,\beta}$ . Hence  $ga - tp(a_{\gamma,\beta}, f_\gamma[M_\gamma], N_\gamma) \in S^{bs}(f_\gamma[M_\gamma])$ . Define an injection  $g : [\gamma, \lambda) \rightarrow \lambda$  by  $g(\alpha) := \text{Min}(A_\alpha)$ . For each  $\alpha \in [\gamma, \lambda)$ ,  $cd(\gamma, \beta) \in A_\alpha$ . So  $g(\alpha) < cd(\gamma, \beta)$ , (otherwise by (7)  $a_{\gamma,\beta} \in f_{\alpha+1}[M_{\alpha+1}] \subset f_\lambda[M_\lambda]$ , but  $a_{\gamma,\beta} = c \notin f_\lambda[M_\lambda]$ ), and  $g$  is an injection from  $[\gamma, \lambda)$  to  $cd(\gamma, \beta)$  which is impossible. Thus (\*) implies a contradiction.  $\dashv$

### 3. NON-FORKING AMALGAMATION

*Hypothesis 3.1.*  $\mathfrak{s}$  is an almost good  $\lambda$ -frame.

In this section we present a theorem from [JrSh 875], which says that we can derive a non-forking relation on models, from the non-forking relation on elements. First we have to define the conjugation property.

#### Definition 3.2.

(1) Let  $p = ga - tp(a, M, N)$ . Let  $f$  be an isomorphism of  $M$  (i.e.  $f$  is an injection with domain  $M$ , and the relations and functions on  $f[M]$  are defined such that  $f : M \hookrightarrow f[M]$  is an isomorphism). Define  $f(p) = ga - tp(f(a), f[M], f^+[N])$ , where  $f^+$  is an extension of  $f$  (and the relations and functions on  $f^+[N]$  are defined such that  $f^+ : N \hookrightarrow f^+[N]$  is an isomorphism).

(2) Let  $p_0, p_1$  be types,  $n < 2 \rightarrow p_n \in S(M_n)$ . We say that  $p_0, p_1$  are *conjugate* if there is an isomorphism  $f : M_0 \hookrightarrow M_1$  such that  $f(p_0) = p_1$ .

#### Claim 3.3.

- (1) In Definition 3.2,  $f(p)$  does not depend on the choice of  $f^+$ .
- (2) The conjugation relation is an equivalence relation.

*Proof.* Easy.  $\dashv$

**Definition 3.4.** Let  $\mathfrak{s}$  be an almost good  $\lambda$ -frame.  $\mathfrak{s}$  is said to satisfy the *conjugation property*, when: if  $p \in S^{bs}(M_1)$  does not fork over  $M_0$ , then there is an isomorphism  $f : M_1 \rightarrow M_0$  such that  $f(p) = p \upharpoonright M_0$ .

**Remark 3.5.** If  $\mathfrak{s}$  satisfies the conjugation property, then  $K_\lambda$  is categorical.

Now we present the properties that a non-forking relation should satisfy.

**Definition 3.6.** Let  $NF \subseteq {}^4K_\lambda$  be a relation. We say  $\bigotimes_{NF}$  when the following axioms are satisfied:

- (a) If  $NF(M_0, M_1, M_2, M_3)$ , then  $n \in \{1, 2\} \rightarrow M_0 \preceq M_n \preceq M_3$  and  $M_1 \cap M_2 = M_0$ .
- (b) The monotonicity axiom: if  $NF(M_0, M_1, M_2, M_3)$  and  $N_0 = M_0, n < 3 \rightarrow N_n \preceq M_n \wedge N_0 \preceq N_n \preceq N_3, (\exists N^*)[M_3 \preceq N^* \wedge N_3 \preceq N^*]$ , then  $NF(N_0, N_1, N_2, N_3)$ .
- (c) The existence axiom: for every  $N_0, N_1, N_2 \in K_\lambda$ , if  $l \in \{1, 2\} \rightarrow N_0 \preceq N_l$  and  $N_1 \cap N_2 = N_0$ , then there is  $N_3$  such that  $NF(N_0, N_1, N_2, N_3)$ .
- (d) The uniqueness axiom: suppose for  $x = a, b$  we have  $NF(N_0, N_1, N_2, N_3^x)$ . Then there is a joint embedding of  $N^a, N^b$  over  $N_1 \cup N_2$ .
- (e) The symmetry axiom:  $NF(N_0, N_1, N_2, N_3) \leftrightarrow NF(N_0, N_2, N_1, N_3)$ .
- (f) The long transitivity axiom: for  $x = a, b$ , let  $\langle M_{x,i} : i \preceq \alpha^* \rangle$  be an increasing continuous sequence of models in  $K_\lambda$ . Suppose  $i < \alpha^* \rightarrow NF(M_{a,i}, M_{a,i+1}, M_{b,i}, M_{b,i+1})$ . Then  $NF(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$ .

**Definition 3.7.** Let  $NF$  be a relation such that  $\bigotimes_{NF}$ . We say that  $NF$  respects the frame  $\mathfrak{s}$  when: if  $NF(M_0, M_1, M_2, M_3)$  and  $ga - tp(a, M_0, M_1) \in S^{bs}(M_0)$ , then  $ga - tp(a, M_2, M_3)$  does not fork over  $M_0$ .

**Theorem 3.8.** *Suppose:*

- (1)  $K$  is categorical in  $\lambda$ .
- (2)  $\mathfrak{s}$  is an almost good  $\lambda$ -frame which satisfies the conjugation property.
- (3)  $I(\lambda^{++}, K) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$ .
- (4)  $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ .
- (5) The ideal  $WDmId(\lambda^+)$  is not saturated in  $\lambda^{++}$ .

Then there is a relation  $NF$  such that  $\bigotimes_{NF}$  and  $NF$  respects the frame  $\mathfrak{s}$ .

*Proof.* By [JrSh 875]: by Corollary [JrSh 875, 4.18],  $K^{3,uq}$  is dense with respect to  $\preceq_{bs}$ . Hence by Theorem [JrSh 875, 5.15], there is a unique relation,  $NF$ , with  $\bigotimes_{NF}$ . Now see Definition [JrSh 875, 5.3].  $\dashv$

#### 4. A FULL GOOD $\lambda$ -FRAME

*Hypothesis 4.1.*  $\mathfrak{s}$  is an almost good  $\lambda$ -frame which satisfies the conjugation property.

**Definition 4.2.**  $nf^{NF} := \{(M_0, M_1, a, M_3) : M_0, M_1, M_3 \in K_\lambda, M_0 \preceq M_1 \preceq M_3, a \in M_3 - M_1 \text{ and for some } M_2 \in K_\lambda, M_0 \preceq M_2, a \in M_2 - M_0 \text{ and } NF(M_0, M_1, M_2, M_3)\}$ .

The following theorem is similar to Claim [Sh:h, 9.5.2].III.

**Theorem 4.3.** *Let  $\mathfrak{s}$  be an almost good  $\lambda$ -frame which satisfies the conjugation property. Then  $\mathfrak{s}^{NF} = (K, \preceq, S^{na}, nf^{NF})$  is a full good  $\lambda$ -frame.*

*Proof.* We will prove the conditions in Definition 2.1:

1. Trivial.
2. (a),(b),(c) are trivial. (d) (basic stability) is satisfied by Theorem 2.6(3).

3. (a) is trivial.

(b) is OK by the monotonicity of  $NF$ , i.e. Definition 3.6(b).

Axiom (c) (local character) is the heart of the matter. Let  $j$  be a limit ordinal, let  $\langle N_i : i \preceq j+1 \rangle$  be an increasing continuous sequence of models in  $K_\lambda$  and let  $p =: ga - tp(c, N_j, N_{j+1}) \in S^{na}(N_j)$ . We have to find  $i < j$  such that  $p$  does not fork over  $N_i$  in the sense of  $nf^{NF}$ , i.e.  $nf^{NF}(N_i, c, N_j, N_{j+1})$ . It is enough to find an increasing continuous sequence  $\langle M_i : i \preceq j \rangle$  such that for each  $i \preceq j$ ,  $N_i \preceq M_i$  and  $NF(N_i, N_{i+1}, M_i, M_{i+1})$  (so  $NF(N_i, N_j, M_i, M_j)$ ) and  $N_{j+1} \preceq M_j$  (for some  $i < j \in M_i$ , so  $nf^{NF}(N_i, c, N_j, N_{j+1})$ ). Without loss of generality,  $cf(j) = j$ . We try to construct  $\langle N_{\alpha,i} : i \preceq j+1 \rangle$  by induction on  $\alpha \in \lambda^+$ , such that:

- (1) For each  $\alpha \in \lambda^+$ ,  $\langle N_{\alpha,i} : i \preceq j+1 \rangle$  is an increasing continuous sequence of models in  $K_\lambda$ .
- (2) For each  $i \preceq j$ ,  $\langle N_{\alpha,i} : \alpha < \lambda^+ \rangle$  is an  $\prec^*$ -increasing continuous sequence of models in  $K_\lambda$  and  $N_{\alpha,j+1} \preceq N_{\alpha+1,j+1}$ .
- (3)  $N_{0,i} = N_i$ .
- (4) For each  $i < j$  and  $\alpha < \lambda^+$ , we have  $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}, N_{\alpha+1,i+1})$ .
- (5) For each  $\alpha \in S =: \{\delta \in \lambda^+ : cf(\delta) = j\}$ , we have  $N_{\alpha,j+1} \cap N_{\alpha+1,j} \neq N_{\alpha,j}$ .

If we succeed, then by clauses (2) and (5), the quadruple

$$\langle N_{\alpha,j} : \alpha < \lambda^+ \rangle, \langle N_{\alpha,j+1} : \alpha < \lambda^+ \rangle, \langle id_{N_{\alpha,j}} : \alpha < \lambda^+ \rangle, S$$

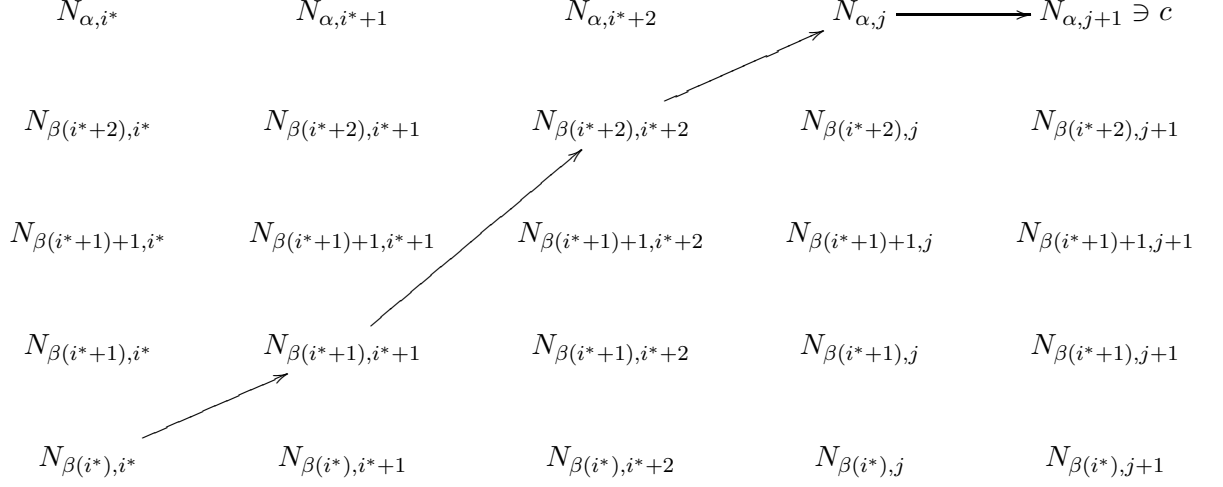
forms a counterexample to Claim 1.3, so it is impossible to carry out this construction.

*Where will we get stuck?* For  $\alpha = 0$ , we will not get stuck, see item (3).

For  $\alpha$  limit, just (1),(2) are relevant, and we just have to take unions and use smoothness.

So we will get stuck at some successor ordinal. Suppose we have defined  $\langle N_{\alpha,i} : i \preceq j+1 \rangle$ . Can we find  $\langle N_{\alpha+1,i} : i \preceq j+1 \rangle$ ? If  $\alpha \notin S$ , then it is easier, so assume  $\alpha \in S$ . Let  $\langle \beta(i) : i \preceq j+1 \rangle$  be an increasing continuous sequence of ordinals such that  $\beta(j) = \alpha$ . If  $N_{\alpha,j} = N_{\alpha,j+1}$ , then we can define  $M_i := N_{\alpha,i}$  and the local character is proved ( $N_j \preceq N_{\alpha,j} = M_j$ , so see the beginning of the proof). So without loss of generality,  $N_{\alpha,j+1} \neq N_{\alpha,j}$ .

In the following diagram, the arrows describe the  $\prec^*$ -increasing continuous sequence  $\langle N_{\beta(i),i} : i \preceq j \rangle \frown \langle N_{\alpha,j+1} \rangle$ . A model that appears at the right and above another model is bigger than it.



By weak local character, there is an element  $c$  and an ordinal  $i^*$  such that  $ga - tp(c, N_{\alpha,j}, N_{\alpha,j+1})$  does not fork over  $N_{\beta(i^*),i^*}$ .

By Definition 2.1(b) (the monotonicity axiom),  $ga - tp(c, N_{\alpha,j}, N_{\alpha,j+1})$  does not fork over  $N_{\alpha,i^*+1}$  and so  $ga - tp(c, N_{\alpha,i^*+1}, N_{\alpha,j+1}) \in S(N_{\alpha,i^*+1})$ . So there is an increasing continuous sequence  $\langle N_{\alpha+1,i}^{temp} : i \preceq j \rangle$  such that for  $i < j$  we have  $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}^{temp}, N_{\alpha+1,i+1}^{temp})$ , and there is  $a \in N_{\alpha+1,i^*+1}^{temp}$  such that  $ga - tp(a, N_{\alpha,i^*+1}, N_{\alpha+1,i^*+1}^{temp}) = ga - tp(c, N_{\alpha,i^*+1}, N_{\alpha,j+1})$ . [Why? For  $i \preceq i^*$  define  $N_{\alpha+1,i}^{temp} = N_{\alpha,i}$ . Choose  $N_{\alpha+1,i^*+1}^{temp}$  which is isomorphic to  $N_{\alpha,j+1}$  over  $N_{\alpha,i^*+1}$  and  $N_{\alpha+1,i^*+1}^{temp} \cap N_{\alpha,j+1} = N_{\alpha,i^*+1}$ . For  $i \in (i^*+1, j]$  choose  $N_{\alpha+1,i+1}^{temp}$  such that  $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}^{temp}, N_{\alpha+1,i+1}^{temp})$ . If  $i$  is limit, then define  $N_{\alpha+1,i}^{temp} := \bigcup \{N_{\alpha+1,\varepsilon} : \varepsilon < i\}$ . Now by the long transitivity of  $NF$  we have  $NF(N_{\alpha,i^*+1}, N_{\alpha,j}, N_{\alpha+1,i^*+1}^{temp}, N_{\alpha+1,j}^{temp})$  and so since  $NF$  respects  $s$ , the type  $ga - tp(a, N_{\alpha,j}, N_{\alpha+1,j}^{temp})$  does not fork over  $N_{\alpha,i^*+1}$ . So by Definition 2.1(e), (the uniqueness of the non-forking extension),  $ga - tp(a, N_{\alpha,j}, N_{\alpha+1,j}^{temp}) = ga - tp(c, N_{\alpha,j}, N_{\alpha,j+1})$ . Hence by the definition of the equality between types, without loss of generality, there is a model  $N_{\alpha+1,j+1}$  such that  $N_{\alpha,j+1} \preceq N_{\alpha+1,j+1}$ , there is an embedding  $f : N_{\alpha+1,j}^{temp} \hookrightarrow N_{\alpha+1,j+1}$  over  $N_{\alpha,j}$  and  $f(a) = c$ . Now for  $i \preceq j$  define  $N_{\alpha+1,i} := f[N_{\alpha+1,i}^{temp}]$ . Why is (5) satisfied?  $c \in N_{\alpha,j+1} \cap N_{\alpha+1,i+1} - N_{\alpha,i+1}$ . By (4) and the long transitivity of  $NF$ , we have  $NF(N_{\alpha,i+1}, N_{\alpha,j}, N_{\alpha+1,i+1}, N_{\alpha+1,j})$ , so  $c \notin N_{\alpha,j}$ , but since  $N_{\alpha+1,i+1} \subset N_{\alpha+1,j}$  we have  $c \in N_{\alpha+1,j}$ . Hence  $c \in N_{\alpha,j+1} \cap N_{\alpha+1,j} - N_{\alpha,j}$ . Hence we can carry out the construction.

(d) Uniqueness: suppose for  $n < 2$ ,  $ga - tp(a^n, M_0, M_1^n)$  does not depend on  $n$ , and  $NF(M_0, M_2, M_1^n, M_3^n)$ , see the diagram below. We have to prove



that  $ga - tp(a^n, M_2, M_3^n)$  does not depend on  $n$ . By the definition of the equality between types, there is an amalgamation  $f^0, f^1, M_1$  of  $M_1^0, M_1^1$  over  $M_0$ . So there are models  $M_3^{n,+}$  and embeddings  $f_n^+ : M_3^n \hookrightarrow M_3^{n,+}$ , such that for  $n < 2$  we have  $NF(f_n[M_1^n], f_n^+[M_3^n], M_1, M_3^{n,+})$  and  $f_n \subset f_n^+$ . Since  $M_2 \cap M_1^n = M_0$ , without loss of generality,  $f_n^+ \upharpoonright M_2 = id_{M_2}$  (we can change the names of the elements in  $M_2 - M_0$ , i.e.  $M_2 - M_1^n$ ). By the long transitivity axiom of  $NF$ , we have  $NF(M_0, M_2, M_1, M_3^{n,+})$ . So by the uniqueness of  $NF$ , there is a joint embedding  $g^0, g^1, M_3$  of  $M_3^{0,+}, M_3^{1,+}$  over  $M_1 \cup M_2$ . So  $g^0 \circ f_0^+, g^1 \circ f_1^+, M_3$  is an amalgamation of  $M_3^0, M_3^1$  over  $M_2$ . Since  $a_n \in M_1^n$ ,  $(g^n \circ f_n^+)(a_n) = f_n(a_n)$  and so it does not depend on  $n$  (since  $f_0, f_1$  are witnesses for  $ga - tp(a_1, M_0, M_1^n)$  does not depend on  $n$ ). So  $ga - tp(a^n, M_2, M_3^n)$  does not depend on  $n$ .

$$\begin{array}{ccccc}
 M_1 & \longrightarrow & M_3^{n,+} & \xrightarrow{g^n} & M_3 \\
 f_n \uparrow & & f_n^+ \uparrow & & \\
 M_1^n & \longrightarrow & M_3^n & & \\
 \uparrow & & \uparrow & & \\
 M_0 & \longrightarrow & M_2 & & 
 \end{array}$$

(e) Symmetry: by the symmetry of  $NF$ , i.e. Definition 3.6(e).

(f) By the corresponding axiom of  $NF$ , i.e. Definition 3.6(c).

(g) Continuity: it is easy to see that continuity follows by local character, because by definition,  $s^{NF}$  is full.  $\dashv$

Now we can present the main theorem: we get a good  $\lambda$ -frame.

**Theorem 4.4.** *Let  $(K, \preceq)$  be an AEC such that:*

- (1)  $K$  is categorical in  $\lambda, \lambda^+$  and  $1 \leq I(\lambda^{+2}, K) < \mu_{unif}(\lambda^{+2}, 2^{\lambda^+})$ .
- (2)  $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ , and  $WDmId(\lambda^+)$  is not saturated in  $\lambda^{+2}$ .

Then:

*there is an almost good  $\lambda$ -frame,  $\mathfrak{s}$  with complete...  $(K_{\mathfrak{s}}, \preceq_{\mathfrak{s}}) = ((K_\lambda, \preceq)$  and a type is basic if it is minimal. Moreover, if  $\mathfrak{s}$  satisfies the conjugation property, then there is a good  $\lambda$ -frame with  $(K_s, \preceq_s) = ((K, \preceq)$ .*

**Remark 4.5.** Background on Weak Diamond appears in [DS] and in Chapter 13 of [Gr:book]. Concerning  $\mu_{unif}(\mu^+, 2^\mu)$ , see the last chapter of [Sh:h], [JrSh 875] or [JrSh 966]. It is "almost  $2^{\mu^+}$ ":  $1 < \mu_{unif}(\mu^+, 2^\mu)$ , If  $\beth_\omega \preceq \mu$ , then  $\mu_{unif}(\mu^+, 2^\mu) = 2^{\mu^+}$  and in any case it is not clear if  $\mu_{unif}(\mu^+, 2^\mu) < 2^{\mu^+}$  is consistent. There are more claims which say that it is a "big cardinal".

*Proof.* By Theorem [Sh E46, 0.2] there is such an almost good frame. So by Theorem 4.3 we have the "moreover".  $\dashv$

While in [Sh:h].II we obtained a good  $\lambda^+$ -frame, here we obtained a  $\lambda$ -good frame. Why is this important? In Section 1 of [Sh:h].III, Shelah defined weakly dimensionality of a good frame, and proved that it is equal to the categoricity in the successor cardinal. Since here we assume categoricity in  $\lambda^+$ , the good  $\lambda$ -frame we obtained here is weakly dimensional.

## 5. THE FUNCTION $\lambda \rightarrow I(\lambda, K)$ IS NOT ARBITRARY

In this section, we prove, under set theoretical assumptions, that there is no AEC,  $(K, \preceq)$ , which is categorical in  $\lambda, \lambda^+ \dots \lambda^{+(n-1)}$ , but has no model of cardinality  $\lambda^{+n}$ . The main results of Section 4 enables to prove only a weaker version of this theorem. But we can prove this theorem, using results of [Sh E46] and [Sh:h].II.

By the last section in [Sh:h].II (alternatively, see Corollary [JrSh 875, 12.6]):

**Fact 5.1.** *Suppose:*

- (1)  $n < \omega$ ,
- (2)  $\mathfrak{s} = (K, \preceq, S^{bs}, nf)$  is a good  $\lambda$ -frame ,
- (3) For each  $m < n$ ,  $I(\lambda^{+(2+m)}, K) < \mu_{unif}(\lambda^{+(2+m)}, 2^{\lambda^{+(1+m)}})$ ,
- (4)  $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < \dots < 2^{\lambda^{+(1+n)}}$ ,
- (5) For each  $m < n$ , the ideal  $WDmId(\lambda^{+1+m})$  is not saturated in  $\lambda^{+(2+m)}$ .

Then there is a model in  $K$  of cardinality  $\lambda^{+(2+n)}$ .

By the following theorem, if  $f : card \rightarrow card$  is a class function (from the cardinals to the cardinals) with  $f(\lambda) = f(\lambda^+) = \dots f(\lambda^{+(n-1)}) = 1$  and  $f(\lambda^{+n}) = 0$ , then under specific set theoretical assumptions (clauses (4),(5), below),  $f$  cannot be the spectrum of categoricity of any AEC.

**Theorem 5.2.** *There are no  $K, \preceq, n, \lambda$  such that*

- (1)  $n \geq 3$  is a natural number,
- (2)  $(K, \preceq)$  is an AEC,
- (3)  $K$  is categorical in  $\lambda^{+m}$  for each  $m < n$ , but  $K_{\lambda^{+n}} = \emptyset$ ,
- (4)  $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < \dots < 2^{\lambda^{+(n-1)}}$ ,
- (5) For each  $m < n - 2$ ,  $WDmId(\lambda^{+1+m})$  is not saturated in  $\lambda^{+(2+m)}$ .

Before we prove Theorem 5.2, we prove a weaker version of it:

**Proposition 5.3.** *The same as Theorem 5.2, but here we assume, in addition, that if  $M_0 \preceq M_1 \preceq M_2$ ,  $a \in M_2 - M_1$  and  $ga - tp(a, M_0, M_2)$  is minimal, then the types  $ga - tp(a, M_1, M_2), ga - tp(a, M_0, M_2)$  are conjugate.*

*Proof.* By Theorem 4.4, there is a good  $\lambda$ -frame with  $(K_s, \preceq_s) = ((K, \preceq))$ . Hence by Fact 5.1, there is a model in  $K$  of cardinality  $\lambda^{+(n)}$ .  $\dashv$

**Remark 5.4.** To our opinion, by Claim [Sh E46, 7.4](p. 76), it is reasonable to assume that  $\mathfrak{s}$  satisfies the conjugation property.

Now we prove Theorem 5.2:

*Proof.* By Theorem [Sh:h, II.3.7](p. 297), there is a good  $\lambda^+$ -frame,  $\mathfrak{s}$  such that its AEC is  $(K, \preceq)$ . Now use Fact 5.1, where  $\lambda^+$  stands for  $\lambda$ .  $\dashv$

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